MATH4060 Assignment 2

Ki Fung, Chan

March 2, 2021

1. Find the order of growth ρ of the following entire functions.

   (a)
   $$f(z) = P(z)e^{Q(z)},$$
   where P and Q are polynomials of degree p and q respectively.

   (b)
   $$e^{e^z}.$$

   (c)
   $$\cos z^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{(2n)!}.$$

Proof. With A, B, C understood to be suitable real positive constants.

(a) For any $r > q$, we have

$$|f(z)| = |P(z)| \cdot |e^{Q(z)}| \leq A_1 e^{B_1|z|^{r-q}} \cdot A_2 e^{B_2|z|^q} \leq Ae^{B|z|^r}$$

so $\rho \leq q$.
If $p = -\infty$ (i.e. $P(z) \equiv 0$), then $\rho = -\infty$.
Now suppose P is not identically zero, then $|P(z)| \geq C$ for some positive constant C for all z with $|z|$ large. By modifying $P(z)$ (and the constant C), we may assume $Q(z) = z^q + a_1 z^{q-1} + \cdots + a_q$. We have

$$|f(z)| = |P(z)| \cdot |e^{Q(z)}| \geq Ce^{\mathrm{Re}(Q(z))}.$$

If we take $z = t$ to be positive real number, we have that for t large,

$$\mathrm{Re}(Q(t)) \geq \frac{1}{2}t^q.$$

Whence for t large enough,

$$|f(t)| \geq Ce^{\frac{1}{2}t^q}$$

Thus for any $r < q$, we have

$$\lim_{t\to\infty} \frac{e^{t^r}}{|f(t)|} = 0.$$

We see that $\rho \geq q$, hence $\rho = q$

(b) For any $r > 0$, we have

$$\lim_{t\to\infty} \frac{e^{t^r}}{|f(t)|} = \lim_{t\to\infty} e^{t^r - e^t} = 0.$$

Whence $\rho = \infty$.

(c)

$$\begin{aligned} |\cos z^{\frac{1}{2}}| \leq & \sum_{n=0}^{\infty} \frac{|z|^n}{(2n)!} \\ \leq & \sum_{n=0}^{\infty} \frac{(|z|^{\frac{1}{2}})^n}{n!} \\ = & e^{|z|^{\frac{1}{2}}}. \end{aligned}$$

Therefore, $\rho \leq \frac{1}{2}$. On the other hand, it can be see easily that $(n\pi + \frac{1}{2})^2$ are zeroes of f for any integer n. But

$$\sum_{n=1}^{\infty} \frac{1}{((n\pi + \frac{1}{2})^2)^{\frac{1}{2}}} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n + \frac{1}{2}} = \infty$$

We see that $\rho \geq \frac{1}{2}$, hence $\rho = \frac{1}{2}$.

□

2. Prove that there exists constant $C > 0$ such that

$$\left| \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| \leq 1 + C \sum_{n=1}^{\infty} \frac{|y|}{y^2 + n^2}.$$

for all $z = x + iy$ with $|x| \leq \frac{1}{2}$ and $|y| \geq 1$.

Proof. First of all, $|z| \geq |y| \geq 1$, so

$$\frac{1}{|z|} \leq 1.$$

Also,

$$\begin{aligned} |z^2 - n^2|^2 = & (x^2 - y^2 - n^2)^2 + (2xy)^2 \\ \geq & (y^2 + n^2 - \frac{1}{4})^2 \\ \geq & \frac{1}{4}(y^2 + n^2)^2. \end{aligned}$$

Finally, we have $2|z| \leq 1 + 2|y| \leq 3|y|$. Therefore,

$$\left| \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| \leq 1 + 6 \sum_{n=1}^{\infty} \frac{|y|}{y^2 + n^2}.$$

□

3. Show that if τ is fixed with $\mathrm{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 as a function of z.

Proof. Let ρ be the order of growth. Let $t = \mathrm{Im}(\tau) > 0$. We have,

$$|\Theta(z|\tau)| \leq \sum_{n=-\infty}^{\infty} e^{\pi(-n^2 t + 2n|z|)}.$$

Note that,

$$\sum_{n=-\infty}^{0} e^{\pi(-n^2 t + 2n|z|)} \leq \sum_{n=-\infty}^{0} e^{\pi(-n^2 t)} = C_1.$$

Next, note that

$$-n^2 t + 2n|z| \leq \frac{1}{2} n^2 t$$

for $|n| \geq \frac{4|z|}{3t}$, so

$$\sum_{n \geq \frac{4|z|}{3t}} e^{\pi(-n^2 t + 2n|z|)} \leq \sum_{n=1}^{\infty} e^{\pi(-\frac{1}{2} n^2 t)} = C_2.$$

Therefore,

$$\begin{aligned} |\Theta(z|\tau)| &\leq C + \sum_{n=1}^{\lfloor \frac{4|z|}{3t} \rfloor} e^{\pi(-n^2 t + 2n|z|)} \\ &\leq C + \sum_{n=1}^{\lfloor \frac{4|z|}{3t} \rfloor} e^{2n\pi|z|} \\ &\leq C + \frac{4|z|}{3t} e^{\frac{8\pi}{3t}|z|^2}. \end{aligned}$$

Hence, $\rho \leq 2$. Finally, it can be see easily (you may find a proof in tutorial 2) that

$$\Theta(z + n\tau|\tau) = e^{-\pi i n^2 \tau} \Theta(z|\tau).$$

hence $\rho \geq 2$ provided we can find some z so that $\Theta(z|\tau) \neq 0$. Its existence can be seen by showing the Fourier coefficients of Jacobi Theta functions are nonzero, for example

$$\int_0^1 \Theta(t|\tau) dt = 1 \neq 0.$$

□

4. Find the Hadamard products for:

(a) $e^z - 1$

(b) $\cos(\pi z)$

Proof. We will make use of the Hadamard product for $\sin z$.

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right) = z \prod_{n\neq 0} E_1\left(\frac{z}{n\pi}\right)$$

(a)

$$\begin{aligned} e^z - 1 &= -2ie^{\frac{z}{2}} \sin(i\frac{z}{2}) \\ &= -2ie^{\frac{z}{2}}(i\frac{z}{2}) \prod_{n\neq 0} E_1\left(\frac{iz}{2n\pi}\right) \\ &= ze^{\frac{z}{2}} \prod_{n\neq 0} E_1\left(\frac{z}{2n\pi i}\right) \end{aligned}$$

(b)

$$\begin{aligned} \cos(\pi z) &= \frac{\sin(2\pi z)}{2\sin z} \\ &= \frac{2\pi z \prod_{n\neq 0} E_1(\frac{2z}{n})}{2\pi z \prod_{n\neq 0} E_1(\frac{z}{n})} \\ &= \prod_{n\in\mathbb{N}} E_1\left(\frac{z}{n+\frac{1}{2}}\right) \end{aligned}$$

□

5. Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

   Proof. Suppose on the contrary that F finitely many zeroes and finite growth order ρ. Then by the Hadamard's theorem,

   $$F(z) = P(z)e^{Q(z)}$$

   for some polynomials P and Q. But then ρ must be an integer by question 1a) □